7 Invariant Subspaces

Invariant Subspaces

- For a linear operator T on $\mathcal{V}$, a subspace $\mathcal{X} \subseteq \mathcal{V}$ is said to be an *invariant subspace* under T whenever $T(\mathcal{X}) \subseteq \mathcal{X}$.
- In such a situation, T can be considered as a linear operator on $\mathcal{X}$ by forgetting about everything else in $\mathcal{V}$ and restricting T to act only on vectors from X. Hereafter, this *restricted operator* will be denoted by $T_{/\mathcal{X}}$.

1. Let T be an arbitrary linear operator on a vector space $\mathcal{V}$. (a) Is the trivial subspace $\{\mathbf{0}\}$ invariant under T? (b) Is the entire space $\mathcal{V}$ invariant under T?

2. Describe all of the subspaces that are invariant under the identity operator I on a space $\mathcal{V}$.

Invariant Subspaces and Matrix Representations Let T be a linear operator on an n-dimensional space $\mathcal{V}$, and let $\mathcal{X}$, $\mathcal{Y}$, ..., $\mathbb{Z}$ be subspaces of $\mathcal{V}$ with respective dimensions r_1, r_2, ..., r_k and bases $\mathcal{B}_\mathcal{X}$, $\mathcal{B}_\mathcal{Y}$, ..., $\mathcal{B}_\mathcal{Z}$. Furthermore, suppose that $\sum_i r_i = n$ and $\mathcal{B} = \mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y} ... \cup \mathcal{B}_\mathcal{Z}$ is a basis for $\mathcal{V}$.

- The subspace $\mathcal{X}$ is an invariant subspace under T if and only if $[T]_\mathcal{B}$ has the block-triangular form

$$[T]_\mathcal{B} = \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix},$$

in which case $A = \left[T_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}} \in \mathrm{Mat}_{r_1 \times r_1}(\mathbb{F})$.

- The subspaces $\mathcal{X}$, $\mathcal{Y}$,..., $\mathcal{Z}$, are all invariant under T if and only if $[T]_\mathcal{B}$ has the block-diagonal form

$$[T]_\mathcal{B} = \begin{pmatrix} A_{r_1\times r_1} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & B_{r_2\times r_2} & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & C_{r_k \times r_k} \end{pmatrix},$$

in which case $A = \left[T_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}}$,

$$B = \left[T_{/\mathcal{Y}}\right]_{\mathcal{B}_\mathcal{Y}}, \quad ..., \quad C = \left[T_{/\mathcal{Z}}\right]_{\mathcal{B}_\mathcal{Z}}.$$

3. Suppose that T is a linear operator on a vector space $\mathcal{V}$. Check is it true that the following three subspaces of $\mathcal{V}$ are T-invariant:

$$\mathrm{im}(T), \qquad \ker(T) \qquad \text{and} \qquad \ker(T - \lambda I)$$

(in last case, under additional assumption that $\dim(\ker(T - \lambda I)) \geq 1$).

4. For $A = \begin{pmatrix} 4 & 4 & 4 \\ -2 & -2 & -5 \\ 1 & 2 & 5 \end{pmatrix}$, $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\boldsymbol{x}_2 = (-1, 2, -1)^\top$, show that the subspace $\mathcal{X}$ spanned by $\mathcal{B} = \{x_1, x_2\}$ is an invariant subspace under A. Then describe the restriction $A_{/\mathcal{X}}$ and determine the coordinate matrix of $A_{/\mathcal{X}}$ relative to $\mathcal{B}$.

5. Let T be a linear operator on an n-dimensional space $\mathcal{V}$, and let $\mathcal{X}$ be subspaces of $\mathcal{V}$ with dimension r and bases $\mathcal{B}_\mathcal{X}$. Show that if the subspace $\mathcal{X}$ is an invariant subspace under T then $[T]_\mathcal{B}$ has the block-triangular form

$$[T]_\mathcal{B} = \begin{pmatrix} A_{r_1 \times r_1} & B \\ \mathbf{0} & C \end{pmatrix},$$

in which case $A = \left[T_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}}$.

Triangular and Diagonal Block Forms When T is an $n \times n$ matrix, the following two statements are true.

- Q is a nonsingular matrix such that

$$Q^{-1}TQ = \begin{pmatrix} A_{r\times r} & B_{r \times q} \\ \mathbf{0} & C_{q\times q} \end{pmatrix}$$

if and only if the first r columns in Q span an invariant subspace under T.

- Q is a nonsingular matrix such that

$$Q^{-1}TQ = \begin{pmatrix} A_{r_1\times r_1} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & B_{r_2\times r_2} & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & C_{r_k \times r_k} \end{pmatrix},$$

if and only if $Q = (Q_1|Q_2|...|Q_k)$ in which Q_i is $n \times r_i$, and the columns of each Q_i span an invariant subspace under T.

6. For $T = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -5 & -16 & -22 \\ 0 & 3 & 10 & 14 \\ 4 & 8 & 12 & 14 \end{pmatrix}$, $\boldsymbol{q}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{q}_2 = (-1, 2, -1, 0)^\top$, verify that $\mathcal{X} = \mathrm{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ is an invariant subspace under T, and then find a nonsingular matrix Q such that

$Q^{-1}TQ$ has the block-triangular form

$$Q^{-1}TQ = \left[\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right]$$

7. Consider again the matrix T and vectors $\boldsymbol{q}_1$ and $\mathbb{Q}_2$ of Exercise 6. Let $\mathcal{B} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4\}$ be a basis for $\mathbb{R}^4$ where

$$\boldsymbol{q}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q}_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Give an answer on the following questions. (a) Are the spaces $\mathcal{X} = \operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ and $\mathcal{Y} = \operatorname{span}\{\boldsymbol{q}_3, \boldsymbol{q}_4\}$ both invariant under T. (b) Is there some invertible matrix Q such that $Q^{-1}TQ$ is block diagonal. If answer is affirmative, find such matrix. (c) If it is possible find $[T_{/\mathcal{X}}]_{\{\boldsymbol{q}_1, \boldsymbol{q}_2\}}$ and $[T_{/\mathcal{Y}}]_{\{\boldsymbol{q}_3, \boldsymbol{q}_4\}}$.

8. Find all subspaces of $\mathbb{R}^2$ that are invariant under

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

> **Remark** In general, scalars λ for which $(A - \lambda I)$ is singular are called the *eigenvalues* of A, and the nonzero vectors in $\ker(A - \lambda I)$ are known as the associated *eigenvectors* for A. As Example 8 indicates, eigenvalues and eigenvectors are of fundamental importance in identifying invariant subspaces and reducing matrices by means of similarity transformations. Eigenvalues and eigenvectors will be discussed at length later.

9. Let T be the linear operator on $\mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3, x_4) =$$
$$= (x_1 + x_2 + 2x_3 - x_4, x_2 + x_4, 2x_3 - x_4, x_3 + x_4)$$

and let $\mathcal{X} = \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the subspace that is spanned by the first two unit vectors in $\mathbb{R}^4$. (a) Explain why $\mathcal{X}$ is invariant under T. (b) Determine $[T_{/\mathcal{X}}]_{\{\boldsymbol{e}_1, \boldsymbol{e}_2\}}$. (c) Describe the structure of $[T]_{\mathcal{B}}$, where $\mathcal{B}$ is any basis obtained from an extension of $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

10. Let T and Q be the matrices

$$T = \begin{pmatrix} -2 & -1 & -5 & -2 \\ -9 & 0 & -8 & -2 \\ 2 & 3 & 11 & 5 \\ 3 & -5 & -13 & -7 \end{pmatrix}, \quad \text{and}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 3 & -4 \\ -2 & 0 & 1 & 0 \\ 3 & -1 & -4 & 3 \end{pmatrix}$$

(a) Explain why the columns of Q are a basis for $\mathbb{R}^4$. (b) Verify that $\mathcal{X} = \operatorname{span}\{Q_{*1}, Q_{*2}\}$ and $\mathcal{Y} = \operatorname{span}\{Q_{*3}, Q_{*4}\}$ are each invariant subspaces under T. (c) Describe the structure of $Q^{-1}TQ$ without doing any computation. (d) Now compute the product $Q^{-1}TQ$ to determine

$$[T_{/\mathcal{X}}]_{\{Q_{*1}, Q_{*2}\}} \quad \text{and} \quad [T_{/\mathcal{Y}}]_{\{Q_{*3}, Q_{*4}\}}.$$

11. (a) Show that if there is a one-dimensional subspace of $\mathcal{V}$ that is invariant under T, then T has a nonzero eigenvector. With another words show that if there is a one-dimensional subspace of $\mathcal{V}$ that is invariant under T, then T has a nonzero $\boldsymbol{x}$ such that $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$.

(b) Let T be a linear operator on $\mathbb{R}^2$ with matrix representation relative to the standard basis given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that the only invariant subspaces of T are $\mathbb{R}^2$ and $\{0\}$.

12. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be a given operator defined with

$$T(a + bt + ct^2) = a + b + c + (a + 3b)t + (a - b + 2c)t^2.$$

Find all one-dimensional subspaces of $\mathcal{P}_2$ that are invariant under T.

13. Let $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be a given linear operator defined with

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} a - b & 4a - 4b \\ -a + 2b + c & b + c \end{pmatrix}$$

Find all one-dimensional subspaces that are invariant under T.

14. Let T be a linear operator on a finite-dimensional vector space $\mathcal{V}$, and let $\mathcal{W}$ be a T-invariant subspace of $\mathcal{V}$. Assume that $\mathcal{B}$ and $\mathcal{B}'$ are basis for $\mathcal{V}$ and $\mathcal{W}$, respectively. Show that the polynomial $g(x) = \det([T_{/\mathcal{W}}]_{\mathcal{B}'} - xI)$ divides the polynomial $p(x) = \det([T]_{\mathcal{B}} - xI)$.

15. **(The Cayley-Hamilton Theorem)** Let T be a linear operator on finite dimensional vector space $\mathcal{V}$, let $\mathcal{B}$ denote a basis of $\mathcal{V}$ and let $f(x) = \det([T]_{\mathcal{B}} - xI)$ be the given polynomial. Show that then

$$f(T) = T_0 \qquad \text{(the zero transformation)}$$

(i.e. $T_0(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathcal{V}$).

InC: 3, 4, 6, 7, 8, 9, 11. HW: 12, 13, 14, 15 + few more problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.